University of New Brunswick
Faculty of Computer Science
CS1303: Discrete Structures
Homework Assignment 2, ***Due Time, Date*** *11:59 PM, February 16, 2021*

Student Name: ____________________________ Matriculation Number: ________________

Instructor: Rongxing Lu
The marking scheme is shown in the left margin and [100] constitutes full marks.

[**40**] 1. For the following 4 statements, please (i) use the logical equivalences $p \to q \equiv \neg p \vee q$ and $p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p)$ to rewrite them without using the symbol $\to$ or $\leftrightarrow$; and (ii) use the logical equivalence $p \vee q \equiv \neg(\neg p \wedge \neg q)$ to rewrite each statement form using only $\wedge$ and $\neg$.

(a) $p \wedge \neg q \to r$

(b) $p \vee \neg q \to r \vee q$

(c) $(p \to r) \leftrightarrow (q \to r)$

(d) $(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$

[**20**] 2. Use truth tables to determine whether the following 4 argument forms are valid or not. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer.

(a)

$$\begin{array}{ll} & p \to q \\ & q \to p \\ \therefore & p \vee q \end{array}$$

(b)

$$\begin{array}{ll} & p \\ & p \to q \\ & \neg q \vee r \\ \therefore & r \end{array}$$

(c)

$$\begin{array}{ll} & p \vee q \\ & p \to \neg q \\ & p \to r \\ \therefore & r \end{array}$$

(d)

$$\begin{array}{ll} & p \wedge q \to \neg r \\ & p \vee \neg q \\ & \neg q \to p \\ \therefore & \neg r \end{array}$$

[**20**] 3. A set of premises and a conclusion are given. Use the valid argument forms listed in Table 1 to deduce the conclusion from the premises, giving a reason for each step. Assume all variables are statement variables.

(a)

a.		$\neg p \vee q \to r$
b.		$s \vee \neg q$
c.		$\neg t$
d.		$p \to t$
e.		$\neg p \wedge r \to \neg s$
f.	$\therefore$	$\neg q$

(b)

a.		$p \vee q$
b.		$q \to r$
c.		$p \wedge s \to t$
d.		$\neg r$
e.		$\neg q \to u \wedge s$
f.	$\therefore$	t

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$, $\sim q$, $\therefore p$ **b.** $p \vee q$, $\sim p$, $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p, $\therefore p \vee q$ **b.** q, $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$, $\therefore p$ **b.** $p \wedge q$, $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to \mathbf{c}$ $\therefore p$

Table 1: Valid Arguments

[**20**] 4. The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. Now, you are visiting the island and have the following encounters with natives.

(a) Two natives A and B address you as follows:
A says: Both of us are knights. B says: A is a knave. What are A and B?

(b) Another two natives C and D approach you but only C speaks.
C says: Both of us are knaves. What are C and D?

(c) You then encounter natives E and F.
E says: F is a knave. F says: E is a knave. How many knaves are there?

Solutions.

[**40**] 1. For the following 4 statements, please (i) use the logical equivalences $p \to q \equiv \neg p \lor q$ and $p \leftrightarrow q \equiv (\neg p \lor q) \land (\neg q \lor p)$ to rewrite them without using the symbol $\to$ or $\leftrightarrow$; and (ii) use the logical equivalence $p \lor q \equiv \neg(\neg p \land \neg q)$ to rewrite each statement form using only $\land$ and $\neg$.

(a) $p \land \neg q \to r$

✓

$$\begin{array}{rcll} p \land \neg q \to r & \equiv & \neg(p \land \neg q) \lor r & \text{by } p \to q \equiv \neg p \lor q \\ & \equiv & \neg(\neg\neg(p \land \neg q) \land \neg r) & \text{by } p \lor q \equiv \neg(\neg p \land \neg q) \\ & \equiv & \neg(p \land \neg q \land \neg r) & \text{by double negation} \end{array}$$

(b) $p \lor \neg q \to r \lor q$

✓

$$\begin{array}{rcll} p \lor \neg q \to r \lor q & \equiv & \neg(p \lor \neg q) \lor (r \lor q) & \text{by } p \to q \equiv \neg p \lor q \\ & \equiv & \neg(\neg\neg(p \lor \neg q) \land \neg(r \lor q)) & \text{by } p \lor q \equiv \neg(\neg p \land \neg q) \\ & \equiv & \neg((p \lor \neg q) \land \neg(r \lor q)) & \text{by double negation} \\ & \equiv & \neg(\neg(\neg p \land \neg\neg q) \land \neg\neg(\neg r \land \neg q)) & \text{by } p \lor q \equiv \neg(\neg p \land \neg q) \\ & \equiv & \neg(\neg(\neg p \land q) \land (\neg r \land \neg q)) & \text{by double negation} \\ & \equiv & \neg(\neg(\neg p \land q) \land \neg r \land \neg q) & \text{by associative laws} \end{array}$$

(c) $(p \to r) \leftrightarrow (q \to r)$

✓

$$\begin{array}{rcl} (p \to r) \leftrightarrow (q \to r) & \equiv & (\neg(p \to r) \lor (q \to r)) \land (\neg(q \to r) \lor (p \to r)) \\ & & \text{by } p \leftrightarrow q \equiv (\neg p \lor q) \land (\neg q \lor p) \\ & \equiv & \neg(\neg\neg(p \to r) \land \neg(q \to r)) \land \neg(\neg\neg(q \to r) \land \neg(p \to r)) \\ & & \text{by } p \lor q \equiv \neg(\neg p \land \neg q) \\ & \equiv & \neg((p \to r) \land \neg(q \to r)) \land \neg((q \to r) \land \neg(p \to r)) \\ & & \text{by double negation} \\ & \equiv & \neg((\neg p \lor r) \land \neg(\neg q \lor r)) \land \neg((\neg q \lor r) \land \neg(\neg p \lor r)) \\ & & \text{by } p \to q \equiv \neg p \lor q \\ & \equiv & \neg(\neg(\neg\neg p \land \neg r) \land \neg\neg(\neg\neg q \land \neg r)) \land \neg(\neg(\neg\neg q \land \neg r) \land \neg\neg(\neg\neg p \land \neg r)) \\ & & \text{by } p \lor q \equiv \neg(\neg p \land \neg q) \\ & \equiv & \neg(\neg(p \land \neg r) \land (q \land \neg r)) \land \neg(\neg(q \land \neg r) \land (p \land \neg r)) \\ & & \text{by double negation.} \end{array}$$

(d) $(p \to (q \to r)) \leftrightarrow ((p \land q) \to r)$

✓

$$\begin{array}{ll}
 & (p \to (q \to r)) \leftrightarrow ((p \land q) \to r) \\
\equiv & (\neg(p \to (q \to r)) \lor ((p \land q) \to r)) \land (\neg((p \land q) \to r) \lor (p \to (q \to r))) \\
 & \text{by } p \leftrightarrow q \equiv (\neg p \lor q) \land (\neg q \lor p) \\
\equiv & \neg(\neg\neg(p \to (q \to r)) \land \neg((p \land q) \to r)) \land \neg(\neg\neg((p \land q) \to r) \land \neg(p \to (q \to r))) \\
 & \text{by } p \lor q \equiv \neg(\neg p \land \neg q) \\
\equiv & \neg((p \to (q \to r)) \land \neg((p \land q) \to r)) \land \neg(((p \land q) \to r) \land \neg(p \to (q \to r))) \\
 & \text{by double negation} \\
\equiv & \neg((\neg p \lor (q \to r)) \land \neg(\neg(p \land q) \lor r)) \land \neg((\neg(p \land q) \lor r) \land \neg(\neg p \lor (q \to r))) \\
 & \text{by } p \to q \equiv \neg p \lor q \\
\equiv & \neg(\neg(\neg\neg p \land \neg(q \to r)) \land \neg\neg(\neg\neg(p \land q) \land \neg r)) \land \neg(\neg(\neg\neg(p \land q) \land \neg r) \land \neg\neg(\neg\neg p \land \neg(q \to r))) \\
 & \text{by } p \lor q \equiv \neg(\neg p \land \neg q) \\
\equiv & \neg(\neg(p \land \neg(q \to r)) \land ((p \land q) \land \neg r)) \land \neg(\neg((p \land q) \land \neg r) \land (p \land \neg(q \to r))) \\
 & \text{by double negation} \\
\equiv & \neg(\neg(p \land \neg(q \to r)) \land p \land q \land \neg r) \land \neg(\neg(p \land q \land \neg r) \land p \land \neg(q \to r)) \\
 & \text{by associative laws} \\
\equiv & \neg(\neg(p \land \neg(\neg q \lor r)) \land p \land q \land \neg r) \land \neg(\neg(p \land q \land \neg r) \land p \land \neg(\neg q \lor r)) \\
 & \text{by } p \to q \equiv \neg p \lor q \\
\equiv & \neg(\neg(p \land \neg\neg(\neg\neg q \land \neg r)) \land p \land q \land \neg r) \land \neg(\neg(p \land q \land \neg r) \land p \land \neg\neg(\neg\neg q \land \neg r)) \\
 & \text{by } p \lor q \equiv \neg(\neg p \land \neg q) \\
\equiv & \neg(\neg(p \land (q \land \neg r)) \land p \land q \land \neg r) \land \neg(\neg(p \land q \land \neg r) \land p \land (q \land \neg r)) \\
 & \text{by double negation} \\
\equiv & \neg(\neg(p \land q \land \neg r) \land (p \land q \land \neg r)) \land \neg(\neg(p \land q \land \neg r) \land (p \land q \land \neg r)) \\
 & \text{by associative laws} \\
\equiv & \neg\mathbf{c} \land \neg\mathbf{c} \text{ by negation laws} \\
\equiv & \mathbf{t} \land \mathbf{t} \text{ by negation of } \mathbf{c} \\
\equiv & \mathbf{t} \text{ by idempotent laws}
\end{array}$$

[**20**] 2. Use truth tables to determine whether the following 4 argument forms are valid or not. Indicate which columns represent the premises and which represent the conclusion, and include a sentence explaining how the truth table supports your answer.

(a)

$$\begin{array}{ll}
 & p \to q \\
 & q \to p \\
\therefore & p \lor q
\end{array}$$

✓

	p	q	$p \to q$	$q \to p$	$p \lor q$
1	T	T	T	T	T
2	F	T	T	F	T
3	T	F	F	T	T
4	F	F	T	T	F
			premise	premise	conclusion

Because the conclusion is false in the critical row (4), the argument is invalid.

(b)

$$
\begin{array}{ll}
 & p \\
 & p \to q \\
 & \neg q \lor r \\
\therefore & r
\end{array}
$$

✓

	p	q	r	$p \to q$	$\neg q$	$\neg q \lor r$
1	T	T	T	T	F	T
2	T	F	T	F	T	T
3	F	T	T	T	F	T
4	F	F	T	T	T	T
5	T	T	F	T	F	F
6	T	F	F	F	T	T
7	F	T	F	T	F	F
8	F	F	F	T	T	T
	premise		conclusion	premise		premise

Because the conclusion is true when all premises are true in all critical rows, i.e., row (1), the argument is valid.

(c)

$$
\begin{array}{ll}
 & p \lor q \\
 & p \to \neg q \\
 & p \to r \\
\therefore & r
\end{array}
$$

✓

	p	q	r	$p \lor q$	$\neg q$	$p \to \neg q$	$p \to r$
1	T	T	T	T	F	F	T
2	T	F	T	T	T	T	T
3	F	T	T	T	F	T	T
4	F	F	T	F	T	T	T
5	T	T	F	T	F	F	F
6	T	F	F	T	T	T	F
7	F	T	F	T	F	T	T
8	F	F	F	F	T	T	T
			conclusion	premise		premise	premise

Because the conclusion is false in the critical row (7) , the argument is invalid.

(d)

$$
\begin{array}{ll}
 & p \land q \to \neg r \\
 & p \lor \neg q \\
 & \neg q \to p \\
\therefore & \neg r
\end{array}
$$

✓

	p	q	r	$p \wedge q$	$\neg r$	$p \wedge q \to \neg r$	$\neg q$	$p \vee \neg q$	$\neg q \to p$
1	T	T	T	T	F	F	F	T	T
2	T	F	T	F	F	T	T	T	T
3	F	T	T	F	F	T	F	F	T
4	F	F	T	F	F	T	T	T	F
5	T	T	F	T	T	T	F	T	T
6	T	F	F	F	T	T	T	T	T
7	F	T	F	F	T	T	F	F	T
8	F	F	F	F	T	T	T	T	F
					conclusion	premise		premise	premise

Because the conclusion is false in the critical row (2), the argument is invalid.

[**20**] 3. A set of premises and a conclusion are given. Use the valid argument forms listed in Table 1 to deduce the conclusion from the premises, giving a reason for each step. Assume all variables are statement variables.

(a)

a.		$\neg p \vee q \to r$	
b.		$s \vee \neg q$	
c.		$\neg t$	
d.		$p \to t$	
e.		$\neg p \wedge r \to \neg s$	
f.	$\therefore$	$\neg q$	

✓

c.		$\neg t$	
d.		$p \to t$	
1.	$\therefore$	$\neg p$	by Modus Tollens

1.		$\neg p$	
a.		$\neg p \vee q \to r$	
	$\equiv$	$\neg(\neg p \vee q) \vee r$	by the representation of if then as or
	$\equiv$	$(p \wedge \neg q) \vee r$	by De Morgan's law
	$\equiv$	$(p \vee r) \wedge (\neg q \vee r)$	by Distributive
	$\equiv$	$(\neg p \to r) \wedge (q \to r)$	by the representation of if then as or
	$\equiv$	$\neg p \to r$ and $q \to r$	by the specification
2.	$\therefore$	r	by Proof by Division into Cases

1.		$\neg p$	
2.		r	
3.	$\therefore$	$\neg p \wedge r$	by Conjunction

3.		$\neg p \wedge r$	
e.		$\neg p \wedge r \to \neg s$	
4.	$\therefore$	$\neg s$	by Modus Ponens

4.		$\neg s$	
b.		$s \vee \neg q$	
f.	$\therefore$	$\neg q$	by Elimination

(b)

a.		$p \vee q$
b.		$q \to r$
c.		$p \wedge s \to t$
d.		$\neg r$
e.		$\neg q \to u \wedge s$
f.	$\therefore$	t

✓

b.		$q \to r$	
d.		$\neg r$	
1.	$\therefore$	$\neg q$	by Modus Tollens

1.		$\neg q$	
e.		$\neg q \to u \wedge s$	
2.	$\therefore$	$u \wedge s$	by Modus Ponens

2.		$u \wedge s$	
3.	$\therefore$	s	by Specialization

a.		$p \vee q$	
1.		$\neg q$	
4.	$\therefore$	p	by Elimination

4.		p	
3.		s	
5.	$\therefore$	$p \wedge s$	by Conjunction

5.		$p \wedge s$	
c.		$p \wedge s \to t$	
5.	$\therefore$	t	by Modus Ponens

[**20**] 4. The logician Raymond Smullyan describes an island containing two types of people: knights who always tell the truth and knaves who always lie. Now, you are visiting the island and have the following encounters with natives.

(a) Two natives A and B address you as follows:

A says: Both of us are knights. B says: A is a knave. What are A and B?

✓

Let PA: A is a knight, and PB: B is a knight.

Assume PA is true,

from "A says: Both of us are knights.", we know $PA \to PA \wedge PB$.

from "B says: A is a knave.", we know $PB \to \neg PA$.

1.		PA	Assumption
2.		$PA \to PA \wedge PB$	
3.	$\therefore$	$PA \wedge PB$, i.e., PA and PB	by Modus Ponens

3.		PB	
4.		$PB \to \neg PA$	
5.	$\therefore$	$\neg PA$	by Modus Ponens

1.		PA	
5.		$\neg PA$	
6.	$\therefore$	$PA \wedge \neg PA = \mathbf{c}$	Contradiction by Negation Law

Therefore, the assumption "PA is true" is wrong, and we know A is a knave, i.e., "$\neg PA$ is true" .

Assume B is a knave, i.e., assume "PB is false" (or $\neg PB$ is true).

from "B says: A is a knave.", we know $\neg PB \to \neg(\neg PA)$, i.e., $\neg PB \to PA$.

7.		$\neg PB$	
8.		$\neg PB \to PA$	
9.	$\therefore$	PA	by Modus Ponens

9.		PA	
10.		$\neg PA$	Above, we have proved $\neg PA$ is true.
6.	$\therefore$	$PA \wedge \neg PA = \mathbf{c}$	Contradiction by Negation Law

Therefore, the assumption "PB is false" is wrong, and we know B is a knight, i.e., "PB is true" .

(b) Another two natives C and D approach you but only C speaks.

C says: Both of us are knaves. What are C and D?

✓

Let PC: C is a knight. and PD: D is a knight.

Assume PC is true,

from "C says: Both of us are knaves.", we know $PC \to \neg PC \wedge \neg PD$.

1.		PC	Assumption
2.		$PC \to \neg PC \wedge \neg PD$	
3.	$\therefore$	$PC \wedge \neg PC = \mathbf{c}$	Contradiction by Negation Law

Therefore, the assumption "PC is true" is wrong, and we know C is a knave, i.e., "$\neg PC$ is true" .

4.	$\neg PC$	
5.	$\neg PC \to \neg(\neg PC \wedge \neg PD)$	
6.	$PC \vee PD$	by De Morgans Laws

6.	$PC \vee PD$	
4.	$\neg PC$	
7.	PD	by Elimination

Therefore, PD is true, and we know D is a knight .

(c) You then encounter natives E and F.

E says: F is a knave. F says: E is a knave. How many knaves are there?

✓

Let PE: E is a knight, and PF: F is a knight.

Assume PE is true and PF is true.

from "E says: F is a knave.", we know $PE \to \neg PF$.

1.		PE	Assumption
2.		$PE \to \neg PF$	
3.		$\neg PF$	by Modus Ponens

4.		PF	Assumption
3.		$\neg PF$	
5.	$\therefore$	$PF \wedge \neg PF = \mathbf{c}$	Contradiction by Negation Law

Therefore, the assumption "PE is true and PF is true." is wrong.

Assume E is knave and F is knave, i.e, $\neg PE$ is true and $\neg PF$ is true.

from "E says: F is a knave.", we know $\neg PE \to PF$.

1.		$\neg PE$	Assumption
2.		$\neg PE \to PF$	
3.		PF	by Modus Ponens

4.		$\neg PF$	Assumption
3.		PF	
5.	$\therefore$	$\neg PF \wedge PF = \mathbf{c}$	Contradiction by Negation Law

Therefore, the assumption "$\neg PE$ is true and $\neg PF$ is true." is wrong. Thus, there is one knave.